On Some Recently Discovered Manuscripts of Thomas Bayes

by

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Abstract

A set of manuscripts attributed to Thomas Bayes was recently discovered among the Stanhope of Chevening papers housed in the Centre for Kentish studies. One group of the manuscripts is directly related to Bayes posthumous paper on infinite series. These manuscripts also show that the paper on infinite series was directly motivated by results in Maclaurin's *A Treatise of Fluxions*. Four other manuscripts in the collection cover a variety of mathematical topics spanning the solution to polynomial equations to infinite series expansions of powers of the arcsine function. It is conjectured that one of latter manuscripts, on the subject of trinomial divisors, predates Bayes's election to the Royal Society in 1742. From the style of writing of the manuscripts, it is speculated that Richard Price, who presented Bayes's now famous essay on inverse probability to the Royal Society, had more of a hand in the printed version of the essay than just the transmission of its contents.

MSC 1991 subject classifications: 01A50, 40A30.

1. Introduction

Thomas Bayes (1701? – 1761) is now famous for his posthumous essay (Bayes 1763a) that gave first expression to what is now known as Bayes Theorem. Bayes's paper was communicated to the Royal Society after his death by his friend Richard Price. Excellent treatments of Bayes's original theorem are given in Stigler (1986) and Dale (1999). His original theorem and approach to probabilistic prediction remained unused and relatively forgotten until Laplace independently rediscovered it and promoted its use later in the 18th century. During his own lifetime, Bayes was probably better known for his work and interests in infinite series. Like his work in probability, his only publication in this area was published posthumously (Bayes, 1763b). Several biographies or biographical sketches of Bayes have been written, the most recent being Bellhouse (2001).

Until recently there have been only four manuscripts of Thomas Bayes that were known to have survived. Three are letters or scientific papers sent to John Canton[1] and the fourth is his notebook.[2] One of the Canton manuscripts is a comment from Bayes to Canton on a paper by Thomas Simpson, which dealt with a special case of the law of large numbers, specifically that the mean of a set of observations is a better estimate of a location parameter than a single observation (Simpson, 1755). Stigler (1986) has discussed this manuscript and has speculated that it was Simpson (1755) that sparked Bayes's interest in probability theory. A second Canton manuscript, written partly in shorthand, is on the subject of electricity. This has been examined by Home (1974-75). The third Canton manuscript is the paper on infinite series that was communicated to the Royal Society by Canton and published posthumously (Bayes, 1763b). The

[1] Royal Society Library, London. Canton Papers, Correspondence, Volume 2, folio 32; and Miscellaneous Manuscripts, Volume I, No. 17.
[2] Notebook. Muniment room, The Equitable Life Assurance Society of London.

notebook has been studied extensively and annotated by Dale (n.d.). Further, Dale (1991) has discussed the early entries in the notebook that are related to the 1763 publication on infinite series.

Recently the author discovered two sets of manuscripts by Bayes, and mostly in the hand of Bayes, that are among the Stanhope of Chevening manuscripts.[3] Philip Stanhope, 2nd Earl Stanhope, had been one of Bayes's sponsors to the Royal Society in 1742. The manuscripts are in two bundles within the Stanhope collection. One bundle, labeled B1 here for easy reference, is entitled by Stanhope, "Mathematical paper of Mr Bayes's communicated Septr 1st 1747." Here is a list of manuscripts in this bundle.

B1.1. A paper in the hand of Bayes showing that an infinite series expansion of the integral of $(z+1/2)\ln[(z+1)/z]$ is divergent.

B1.2. A paper in the hand of Bayes that contains most of the material in Bayes (1763b).

B1.3. A paper in the hand of Bayes deriving Stirling's approximation to $z!$ using a convergent infinite series.

B1.4. A paper in the hand of Bayes to find the reciprocal of a particular infinite series.

B1.5. A recursive method in the hand of Bayes for the infinite series representation of $(\arcsin(z))^n$ for any integer n.

With one exception the papers in this bundle are all related and are relevant to the posthumous publication on infinite series (Bayes, 1763b). The exception in the bundle is a paper on infinite series for powers of the arcsine function (B1.5). The second bundle, labeled here B2 for easy reference, contains a miscellaneous collection consisting of a single letter and some additional unrelated mathematical results in manuscript form, some in Bayes's hand and some in

[3] Centre for Kentish Studies, Maidstone, Kent. U1590/C21: Papers by several eminent Mathematicians addressed to or collected by Lord Stanhope.

Stanhope's hand, mostly copies of Bayes's original results. Here is a description of the contents of B2.

B2.1. A note in the hand of Stanhope attributing to Bayes an infinite series expansion of the first fluxion in terms of finite differences, dated August 12, 1747.

B2.2. A letter from Bayes to Stanhope dated April 25, 1755 concerning a paper by Patrick Murdoch.

B2.3. A paper in the hand of Bayes introducing a box notation peculiar to Bayes and several algebraic derivations related to this notation.

B2.4. A paper in the hand of Stanhope entitled, "The Reverend Mr. Bayes's Paper concerning Trinomial divisors."

B2.5. Several manuscript pages in the hand of Stanhope that appear to be scribblings and algebraic derivations related to B2.4.

B2.6. A note in the hand of Stanhope attributing to Bayes an infinite series expansion of the ratio of first fluxions of two variables to an infinite series expansion in terms of finite differences with some briefs notes on the proof by Stanhope.

These manuscripts provide insight into a number of aspects of Bayes's work and career. One aspect can be dealt with immediately after examining the only letter in the collection (B2.2), the letter from Bayes to Stanhope dated April 25, 1755. It appears that Bayes played the role of critic or commentator for a network of mathematicians that centered on Stanhope or perhaps John Canton. Just as Bayes was providing comments to Canton on Simpson's work he was also commenting to Stanhope on a paper by another mathematician, Patrick Murdoch. From the context of Bayes's letter as well as another by Murdoch in the Stanhope collection, it appears that Stanhope had sent Bayes a copy of Murdoch's paper to look at. Then, after Stanhope had received Bayes's comments, he forwarded them to Murdoch who made his reply through Stanhope. The surviving letter of Bayes is an acknowledgement that Bayes could not agree with Murdoch's response and had enclosed a paper of his own, which has not survived, in response to the response.

Other aspects of Bayes's work, as they relate to the newly discovered manuscripts will be discussed here. As noted already some of the remaining manuscripts in the collection are related to published work on infinite series. This also includes some notebook entries. The manuscripts that are relevant to Bayes (1763b) are discussed in §3. The remaining manuscripts in the collection are discussed in §4. They show that Bayes had a wide variety of mathematical interests, which is also evident in the notebook. The manuscripts give a fairly conclusive starting date as to when Bayes began keeping his notebook. This and some other aspects relating to the dating of the manuscripts themselves are discussed in §2. There is circumstantial evidence in the manuscripts, discussed in §5, supporting Stigler's (1986) conjecture as to when Bayes became interested in probability. Finally the style in which the manuscripts were written leads to some questions, given in §6, as to which parts of Bayes's essay (Bayes, 1763a) on probability are due to Bayes and which are the work of Richard Price.

2. The Manuscripts and Bayes's Notebook

The earliest date in the manuscript collection is in a note (B2.1) written by Stanhope on a scrap of paper that reads:

"Theorem mentioned to me at Tunbridge Wells by M^{r} Bayes Aug. 12. 1747.

$$\dot{y} = \underset{\cdot}{y} - \frac{1}{2}\underset{\cdot\cdot}{y} + \frac{1}{3}\underset{\cdot\cdot\cdot}{y} - \frac{1}{4}\underset{::}{y} + \frac{1}{5}\underset{\therefore}{y} - \frac{1}{6}\underset{\vdots\vdots}{y} + \&c\text{''}$$

The dot over the *y* denotes the fluxion or differential *dy*/*dt* and the number of dots under the *y* denotes the order of differencing in terms of Newton's forward differences. There are two things of interest about this scrap. The first is that this result and a related one,

$$\underset{.}{y} = \dot{y} + \frac{1}{2}\ddot{y} + \frac{1}{2.3}\dddot{y} + \frac{1}{2.3.4}\ddddot{y} \; \& c \, ,$$

which together provide the relationship between derivatives and finite differences, are the very first results that appear in Bayes's undated notebook. The early results in the notebook up to page 10 are related to a form of Stirling's (1730) approximation to *z*! A related manuscript (B1.1) in one of the bundles has written on the side opposite the mathematical paper, "Mathematical paper of M^{r} Bayes's communicated Sept^{r} 1st 1747." Consequently, it may be safely assumed that the notebook dates from August of 1747. The second thing about the scrap (B2.1) is that Bayes did not provide any proofs of the theorems relating differences and derivatives, and never published his results. Bayes also claimed to have obtained the general result, stating in his notebook that, "y^{e} relation between $\ddot{x}$ & $\underset{..}{x}$ & so on may be found". In a sense, Bayes was ahead of his time. The first publication that I can find related to these results is due to Lagrange in 1772 and again in 1792 (Lagrange, 1869-70); see also Goldstine (1977, 164 – 165) for a discussion. Lagrange's result is the general one, giving the left hand side of either equation as a general order of derivative or difference.

The latest date in the manuscripts is in a letter from Bayes to Stanhope (B2.2). It is dated April 25, 1755. Although there are only three dates given in the two bundles, and they cover the years 1747 to 1755, the manuscripts may actually cover a twenty-year interval. In §4.1, I will argue that one of the manuscripts (B2.4) may predate 1742 and is related to Bayes's election to the Royal Society. Another manuscript, all in Stanhope's hand, has written at the bottom of it, “This is a Theorem shewn me by the late M^r Bayes.” Consequently, it is likely that the result was sent to Stanhope late in Bayes’s life.

Most of the manuscripts have related entries in Bayes’s notebooks. There are two exceptions. The first is a manuscript on trinomial divisors (B2.4), discussed in §4.1 that may predate the notebook. The second is the manuscript (B2.6), discussed in §4.4 that refers to the late Mr. Bayes and so may postdate the notebook.

3. Bayes’s Published Work on Infinite Series and Related Manuscript Material

Among the bundle of Bayes’s manuscripts labeled, “Mathematical paper of M^r Bayes’s communicated $Sept^r$ 1^{st} 1747.” is a paper (B1.2), which opens with the statement:

> “It has been asserted by several eminent Mathematicians that the sum of the Logarithms of the numbers 1. 2. 3. 4. 5 &c to z is equal to $\frac{1}{2}\log, c + \overline{z+\frac{1}{2}} \times \log, z$ lessened by the series $z - \frac{1}{12z} + \frac{1}{360z^3} - \frac{1}{1260z^5} + \frac{1}{1680z^7} - \frac{1}{1188z^9} + \&c$ if c denote the circumference of a circle whose radius is unity.”

This quotation is also the second paragraph verbatim of the posthumous publication on infinite series (Bayes, 1763b). The manuscript continues as the paper does, but ends with the sentence:

> “Much less can that series have any ultimate value which is deduced from it by taking $z = 1$ & is supposed to be equal to the logarithm of the square root of the periphery of a circle whose radius is unity.”

The ending is two sentences earlier than the published paper, which concludes with a discussion of the divergence of the series associated both with the sum of logarithms of odd numbers and with the sum of numbers in an arithmetic progression. A discussion of the divergent series and its use related to Stirling’s formula may be found in Tweddle (1988).

An earlier version of Bayes (1763b) is also in the same manuscript bundle (B1.1). This version opens with:

"It is said that the integral of $\text{Log},\overline{\frac{z+1}{z}}\Big|^{z+1/2}$ is $= z - \frac{1}{12z} + \frac{1}{360z^3} - \frac{1}{1260z^5} + \frac{1}{1680z^7} - \&c$ But in the following manner it will evid[tly] appear that this series do's not converge."

The method of proof of divergence of the series is the same. It is shown that at some point subsequent terms in the series begin to increase so that the series diverges. Compared to Bayes (1763b) and the first manuscript (B1.2), this manuscript contains more mathematical detail on how the series diverges.

It is evident from these and other manuscripts in this bundle, as well as from Bayes's notebook, that Bayes was motivated to examine this infinite series by Maclaurin's *A Treatise of Fluxions*, in particular the use of the divergent series in article 842 in Maclaurin (1742) to obtain one of the forms of Stirling approximation to $z!$, namely $\sqrt{2\pi\, z}\, z^z e^{-z}$. Page 3 of the notebook contains results related to the series as well as notes, or partial transcriptions, from articles 827, 839, 842 and 847 of Maclaurin (1742). This has been noted by Dale (1991). The evidence from the manuscript bundle is that one of the manuscripts is a derivation of the Stirling approximation that does not rely on the divergent series in question. In this context the last two sentences of Bayes (1763b) is a commentary on articles 839 and 840 of Maclaurin (1742) in which the sum of logarithms of odd numbers (article 840) and the sum of logarithms of numbers in arithmetic progression (article 839) are obtained, again by using the divergent series.

The dating of the manuscripts by Stanhope to 1747 answers a question first put forward by Deming (1963). Deming noted that the divergence of the series for $z = 1$ was known by Euler (1755) and wondered if Bayes had obtained his insight from Euler's work. Dale (1991), using evidence from Bayes's notebook, concluded that, "there is no evidence of Bayes's being acquainted with Euler's work on the series for log z!". Stanhope's dating confirms Dales's conclusion; the result was obtained probably fifteen years prior to its publication and eight years prior to Euler (1755).

Bayes provided an alternate proof of Stirling's approximation to $z!$ that did not rely on the divergent series. In manuscript B1.3 in the first bundle, Bayes began by assuming the relationship

$$ɨ^{p} = \frac{k\, z^{z} \sqrt{z}}{z!} ,$$

where ɨ is the natural base e (or in the words of Bayes, "Let ɨ be the ratio whose hyperbolic Logarithm is = 1") and k is a constant. Following immediately upon this definition of $ɨ^{p}$, Bayes goes on to show that p is the integral of $(z+1/2)\left(\log(z+1)-\log(z)\right)$, so that an infinite series approximation is relevant. As noted already from manuscripts B1.1 and B1.2 in the first bundle, the infinite series approximation used by Maclaurin and others was not appropriate since that series diverges. In his notebook, Bayes wrote out five infinite series expansions for $(z+1/2)\left(\log(z+1)-\log(z)\right)$ and found the integral of one of them. Dale (1991), who has thoroughly reviewed the section of Bayes's notebook in which these series appear, has given the label S3 to the series for which Bayes obtained the integral. The series S3, as expressed by Dale (1991), is

$$1-\frac{1}{12}\Delta(1/z)+\frac{1}{120.3!}\Delta^{3}(1/z)-\frac{1}{30.5!}\Delta^{5}(1/z)+\cdots ,$$

where Δ is the first finite difference. Bayes's expression for the integral is an infinite series itself. The infinite series derived and used in the manuscript turns out as well to be the same series labeled S3. Using this series Bayes concluded that, "p is always less z & greater than $z - \frac{1}{12z}$ & therefore when z is infinite $p = z$"; and so $z! = k\, z^z \sqrt{z} / Ɨ^{z}$. Bayes then went on to show, by considering the middle term $(2z)!/(z!z!)$ in a binomial expansion and by using Wallis's (1655) infinite product representation of $\pi/2$ (though stated without reference), that k is equal to $\sqrt{2\pi}$ so that Maclaurin's version of Stirling's approximation is obtained.

Another manuscript (B1.4) in the same bundle opens wth:

"To find $\frac{1}{x}$ when $\dot{z} = 1$ & $x = 1 + \frac{z}{2} + \frac{z^2}{2.3} + \frac{z^3}{2.3.4} + \frac{z^4}{2.3.4.5} + \&$"

and continues with a derivation of the result. On noting that the infinite series for x reduces to $(e^z - 1)/z$, it may be seen that finding $1/x$ is the same as finding

$$\frac{1}{x} = \frac{z}{e^z - 1}.$$

Dale (1991) has shown that a related expression

$$\frac{ze^z}{e^z - 1} - \frac{z}{2}$$

is a reasonable function to consider for finding $\log(z!)$. Dale (1991) has studied this latter expression and its relation to results in Bayes's notebook.

4. Other Mathematical Results in the Manuscripts

There is one manuscript in the bundle labeled, "Mathematical paper of M^{r} Bayes's communicated Sept^{r} 1st 1747." that does not fit in with the rest of the bundle. This manuscript (B1.5) deals with infinite series derivations for the arcsine function. It is discussed in §4.1. The second bundle of manuscripts contains a variety of unrelated mathematical results, only one of which has a related result in Bayes's notebook. These are discussed in §§4.2 – 4.4.

4.1 A Recursive Method for the Determination of Power Series for Powers of Arcsines

A manuscript (B1.5) to determine recursively the infinite series for powers of the arcsine (x^n, where $x = \arcsin(z)$) begins,

"If *x* be the arch & *z* the sine the radius being unity.

$$\& \; x^n = z^n + \frac{A\,z^{n+2}}{\overline{n+1} \times n+2} + \frac{B\,z^{n+4}}{\overline{n+1} \times \overline{n+2} \times \&c \times \overline{n+4}} + \frac{C\,z^{n+6}}{\overline{n+1} \times \overline{n+2} \times \&c \times \overline{n+6}} + \&c$$

$$\& \; x^{n-2} = z^{n-2} + \frac{a\,z^{n}}{\overline{n+1} \times n} + \frac{b\,z^{n+2}}{\overline{n-1} \times n \times \&c \times \overline{n+2}} + \frac{c\,z^{n+4}}{\overline{n-1} \times n \times \&c \times \overline{n+4}} + \&c$$

Then $A = a + n^2 \quad B = b + \overline{n+2}|^2 A \quad C = c + \overline{n+4}|^2 B \quad D = d + \overline{n+6}|^2 C$ & so on

Also $$\frac{x^{n-1}}{\sqrt{1-z^2}} = z^{n-1} + \frac{A\,z^{n+1}}{n \times \overline{n+1}} + \frac{\mathrm{B}\,z^{n+3}}{n \times \overline{n+1} \times \&c \times \overline{n+3}} + \frac{\mathrm{C}\,z^{n+5}}{n \times \overline{n+1} \times \&c \times n+5} + \&c\text{''.}$$

There are some minor slips of the pen in what Bayes has written. In Bayes's first equation, $\overline{n+1} \times n+2$ should read $\overline{n+1} \times \overline{n+2}$ or the product of $n + 1$ and $n + 2$, and in the last equation $\& c \times n+5$ should read $\& c \times \overline{n+5}$. Having assumed these general relationships, Bayes goes back to the cases for $n = 1$ and 2. Without proof, he correctly states the infinite series expansions for each of *x*, x^2 and $x/\sqrt{1-z^2}$. He then develops correctly the series expansions for x^3 and x^4 using the relationships quoted above. In both these latter cases Bayes states that he does not notice any regularity in the coefficients for powers of *z*, unlike *x* and x^2, and so he quits without any further comment. No hint is given as to where the general recursion equations came from and no source is given for his initial conditions in the recursion.

Some of the missing details are worked out on page 114 of Bayes's notebook. The infinite series expansion for $x = \arcsin(z)$ is derived by building up a series of fluxional relationships. For example, if $x = \arcsin(z)$, then $\sin(x) = z$ so that $dz/dt = \cos(x)\,dx/dt$. On using the relationship between sines and cosines, the latter expression in the standard fluxional notation is given by $\dot{z} = \sqrt{1 - z^2}\,\dot{x}$. He then derives an equation in z, $\dot{x}$ and $\ddot{x}$, and equations in higher orders of the fluxions of x. From these relationships, and on assuming z = 0, $\dot{x} = 1$ and $\ddot{x} = \dddot{x} = \cdots = 0$, Bayes obtains the infinite series expansion for $x = \arcsin(z)$. From there Bayes considers the relationship $v = x^n$ so that $\dot{v} = n\,x^{n-1}\dot{x}$, and again builds up equations in higher order fluxional relationships. In particular using Bayes's notation, the relationship $\ddot{v} - z^2\ddot{v} - z\,\dot{v} = n \times \overline{n-1}\,x^{n-2}$ proves useful. Bayes begins with the assumption of the series

$$x^n = z^n + \frac{A\,z^{n+2}}{\overline{n+1} \times \overline{n+2}} + \frac{B\,z^{n+4}}{\overline{n+1} \times \overline{n+2} \times \&c \times \overline{n+4}} + \frac{C\,z^{n+6}}{\overline{n+1} \times \overline{n+2} \times \&c \times \overline{n+6}} + \&c$$

as in the manuscript. He uses $\ddot{v} - z^2\ddot{v} - z\,\dot{v} = n \times \overline{n-1}\,x^{n-2}$ to obtain the manuscript expression for x^{n-2} and the relations between A and a, B and b, and so on. Although not given in the notebook the manuscript's infinite series expansion for $x^{n-1} / \sqrt{1 - z^2}$ can be obtained using on taking dx^n / dt and using the relationship $\dot{z} = \sqrt{1 - z^2}\,\dot{x}$.

Bayes's insight into the problem was recognizing a general form for the expansion of x^n. There is not hint in the notebook that would show his reasoning in considering the form of the expansion that he did. Further, his work appears inspired by de Moivre (1730, 109 – 122) in which de Moivre develops infinite series expansions at times using a recursive approach. The notation used by Bayes is the same as de Moivre (1730).

Bayes's manuscript was written sometime after 1750. The material in the notebook that is related to the manuscript appears on page 114. Earlier, on page 86 of the notebook, there is an entry with heading, "Estimate of the National debt upon 31 Dec. 1749". Dale (n.d.) has identified the entry as an extract from pages 150 and 151 of the *London Magazine* for 1750 so that it may be safely assumed that later entries in the notebook postdate 1750.

Without giving any specific references or dates, Bromwich (1926, 197) has attributed the derivations of $\arcsin(z)/\sqrt{1-z^2}$ and $(\arcsin(z))^2$ to Euler. The derivations of the infinite series representations of these functions do not appear explicitly in Euler (1755) and the next reasonable publication for the derivations to appear is Euler (1768) after Bayes's death. Further, Dale (1991) has argued that it is unlikely that Bayes was familiar with Euler (1755) so that, like his work on the divergent series discussed in §3, Bayes's work on infinite series expansions for powers of arcsines is original and probably predates Euler's work in the area.

4.2 Trinomial Divisors

In modern notation Cotes (1722) obtained the relation

$$x^{2n}+1=\prod_{i=1}^{n}\left[x^2-2x\cos\left(\frac{\pi}{2}\,\frac{2i-1}{n}\right)+1\right].$$

Using geometrical arguments Maclaurin (1742), in articles 765 through 768 of his work, generalized this relationship to the trinomial $x^{2n}-2\cos(\theta)x^n+1$ on the left hand side of the equation for any angle $0<\theta<2\pi$. On the right hand side of the equation for the general relationship there is an additional term in θ under the argument for the cosine. Maclaurin's proof involved the examination of a unit circle whose circumference was divided into n arcs of equal length. These arcs were then related to the arc subtended by the angle θ. Earlier, de Moivre (1730) had obtained the generalization using arguments different from Maclaurin. De Moivre's approach involved a form of induction since he built up the general result through looking at cases for $n = 1, 2, \ldots$ and so on.

In the second bundle of manuscripts is a paper (B2.4) in Stanhope's hand entitled, "The Reverend M^{r} Bayes's Paper concerning trinomial divisors." It may be assumed that the paper in the bundle is a transcription of an original work sent by Bayes to Stanhope. The original was either sent back to Bayes or sent on to someone else. Bayes's proof is similar to, yet different from, Maclaurin's (1742) treatment of the problem. Bayes also considers a unit circle with the circumference divided into arcs of equal length and he uses geometrical arguments to obtain his results. Unlike Maclaurin who initially considers *n* arcs, Bayes begins by considering two arcs of equal length. They are obtained from three points on the circumference of the circle, say A, B and C such that the points form an isosceles triangle with lengths of the sides AB and BC equal to *s* and so the arcs subtended by the chords are of equal length. Then he considers another point L on the circumference and sets the lengths of the chords LA = *p*, LB = *q* and LC = *r* and proceeds to show that for the diameter of value 2,

$$(p^2 - 2) + (q^2 - 2) + (r^2 - 2)(s^2 - 2) = 0 .$$

Then Bayes goes to the case of several arcs of equal length and obtains relations similar to the above one. Like de Moivre (1730), Bayes proceeded inductively. He uses his geometric results to show that, if $x^2 + ax + 1$ divides into $x^{2n} + bx^n + 1$ and $x^{2n+2} + cx^{n+1} + 1$, then it also divides $x^{2n+4} + dx^{n+2} + 1$ provided that $b + d + ac = 0$, which is related to the above result for two arcs of equal length. Then Bayes looks at the cases for $n = 1, 2$ and 3.

There is a substantial amount of mathematical scribbling (B2.5) in Stanhope's hand on separate sheets in this manuscript bundle. Some of the material appears related to the result on trinomial divisors in that the scribbling is an attempt on Stanhope's part to algebraically fill in the detail of what Bayes had perhaps considered too trivial to include.

Stanhope did not date the manuscript on trinomial divisors. I would argue that the manuscript predates Maclaurin (1742) and hence Bayes's election to the Royal Society that same year. Had the manuscript been written after Maclaurin's *A Treatise of Fluxions*, it would have been seen as a similar but alternate proof to what appeared in the book. Further, the proof that appears in Maclaurin is easy to follow and practical in its application. Why then would Stanhope copy Bayes's entire proof, label it as Bayes's theorem and file it among his own papers? Further circumstantial evidence is that, although there is a brief reference to Cotes (1722) on page 78 and extracts from Maclaurin (1742) on page 3 of Bayes's manuscript notebook, there is no hint of the theorem in the notebook so that the result may predate the notebook. Bayes was interested in geometrical problems; Dale (n.d.) has noted seven or eight different places in the notebook where topics in geometry or trigonometry are covered. As argued in §2, the notebook probably dates from 1747. If the manuscript predates 1742, then Bayes's nomination certificate presented at a Royal Society on April 8, 1742 may be put into context. The certificate, with Stanhope as the first signatory states in part, "… we propose and recommend him as a Gentleman of known merit, well skilled in Geometry and all parts of Mathematical and Philosophical Learning …"[4] There has been much speculation over the reason for Bayes's election to the Royal Society. His only publication, prior to his election, was an anonymous one ([Bayes], 1736) and most scholars attribute his election to this publication. However, the 1736 work is not about geometry, but instead a treatise on the foundations of fluxions or differential calculus written in reply to Berkeley's (1734) criticisms of the subject. See Jesseph (1993) for a discussion of Berkeley's work as well as Bayes's reply, and Smith (1980) for an alternate view of Bayes's reply. It is very possible that Stanhope sought out Bayes after the publication Bayes's book; Maclaurin, for example, was at Stanhope's home at Chevening in 1743 a year after Maclaurin's own book on fluxions was published (Maclaurin, 1982, 112). After their initial meeting, Bayes and Stanhope corresponded on mathematical matters as the manuscript collection shows. My interpretation is that the theorem was part of their early correspondence leading Stanhope to include "well skilled in Geometry" in his nomination of Bayes to the Royal Society. At least some of Stanhope's

[4] Royal Society Library, London. Cert I, 210. Nomination certificate of Thomas Bayes.

scribblings (B2.5) were notes to himself justifying the validity of Bayes's theorem on trinomial divisors.

4.3 Results Using the Box Notation

On page 73 of his notebook, Bayes introduces a notation to describe the sum of all products of factors in which the sum of the powers of the factors is *n*. In Bayes's words, he begins, "If $\boxed{abcde}^{\,n}$ signify the sums of all y^e factors of y^e n^{th} dimension which can be formed of *a*, *b*, *c*, *d*, *e*". Bayes then goes on to establish a number of relationships depending on the number of letters used and on successive values of *n*. Some of these relations are given below. One of the manuscripts (B2.3) in the second bundle begins in almost exactly the same way. The "If" is replaced by "Let" and the phrase ends with, "… of *a*, *b*, *c*, *d*, *e* &c added together." The manuscript is divided into three sections or articles, of which the first two are the totality of what appears in the notebook and the third is a new result. In both the notebook and B2.3 Bayes begins with what follows from his initial definition of his own notation,

$$\boxed{ab}^{\,n} = a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + b^n$$

$$\boxed{abc}^{\,n} = \boxed{ab}^{\,n} + \boxed{ab}^{\,n-1}c + \boxed{ab}^{\,n-2}c^2 + \cdots + c^n$$

$$\boxed{abcd}^{\,n} = \boxed{abc}^{\,n} + \boxed{abc}^{\,n-1}d + \boxed{abc}^{\,n-2}d^2 + \cdots + d^n$$

He then derives, again in both the notebook and in B2.3, a general relation

$$(\boxed{abcde}^{\,n} - \boxed{bcdef}^{\,n})/(a\text{-}f) = \boxed{abcdef}^{\,n-1}.$$

Bayes goes on to prove the following result, which is not in the notebook. If *a*, *b*, *c* and *d* are the roots of the equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, then

$$\boxed{ab}^{\,n-1} - A\,\boxed{ab}^{\,n-2} + B\,\boxed{ab}^{\,n-3} - C\,\boxed{ab}^{\,n-4} + D\,\boxed{ab}^{\,n-5} = 0,$$

$$\boxed{abc}^{\,n-2} - A\,\boxed{abc}^{\,n-3} + B\,\boxed{abc}^{\,n-4} - C\,\boxed{abc}^{\,n-5} + D\,\boxed{abc}^{\,n-6} = 0, \text{ and}$$

$$\boxed{abcd}^{\,n-3} - A\,\boxed{abcd}^{\,n-4} + B\,\boxed{abcd}^{\,n-5} - C\,\boxed{abcd}^{\,n-6} + D\,\boxed{abcd}^{\,n-7} = 0.$$

There is some discussion in Bayes's notebook on finding roots of polynomial equations. However, there is nothing even remotely resembling this result in the notebook.

4.4 Another Infinite Series

A theorem in the second bundle (B2.6) is stated simply as

$$\frac{\dot{y}}{\dot{x}} = \frac{\underset{\cdot}{y} - \frac{1}{2}\underset{\cdot\cdot}{y} + \frac{1}{3}\underset{\cdots}{y} - \frac{1}{4}\underset{::}{y}}{\underset{\cdot}{x}} + \&c\,.$$

The statement of the theorem is in Stanhope's hand as well as a sketch of the proof. At the bottom of the page, also in Stanhope's hand, is written, "This is a Theorem shewn me by the late M^{r} Bayes." From the way in which the letters are formed with the quill pen, the attribution appears to have been written at a different time than the theorem and proof. It is unclear if the proof belongs to Bayes or just the statement of the theorem. Also the theorem has no context; no reason is given for its consideration.

5. Bayes's Interest in Probability

There has been much speculation as to when Bayes became interested in probability. For example, Barnard (1958) suggested that Bayes had learned his mathematics, and implicitly probability from de Moivre who gave lectures or lessons in a London coffee house. This conjecture was made before it was discovered that Bayes had attended the University of Edinburgh (see, for example, Dale, 1991 and Bellhouse, 2001) and had studied mathematics with James Gregory, nephew of the more eminent mathematician of the same name, in the early 1720s. Stigler (1986) more reasonably has argued that Bayes became interested in probability after reading Simpson (1755), after which he provided comments on Simpson's paper to John Canton.

The only letter of Bayes, dated April 25, 1755, in the Stanhope manuscript collection (B2.2), along with a related letter in the collection from Patrick Murdoch to Stanhope, helps to support Stigler's conjecture. Murdoch's letter shows that he was aware of Stanhope's interest in probability, but at the same time he was unaware of any interest on the part of Bayes. Bellhouse (2001) has provided more detail on the exchange of letters, as well as an earlier letter of Stanhope, that relate to Bayes's probable lack of interest in probability prior to 1755.

6. **The Manuscripts and Bayes's Essay on Probability**

In terms of their content, the manuscripts in the Stanhope collection are unrelated to Bayes's work in probability (Bayes, 1763a). However, the style in which the manuscripts were written may give some insight into Price's role as redactor of the final publication. There are a number of things that can be noted about Bayes's style. First, Bayes has only two related references to the work of others and they are vague: "It has been asserted by several eminent Mathematicians …" and "It is said that the integral of …" Related to references that "should" be there, there is, for example, no reference to Maclaurin (1742) who had obviously inspired Bayes to come up with an alternate derivation to $z!$ that is discussed in §3. This style is evident in Bayes's essay on probability (Bayes, 1763a); references in the paper are obviously due to Price. The remaining observations are not evident in Bayes's essay and lead to the conjecture that Price had more input into Bayes's essay than simply sending it to Canton intact in the form that appears from pages 376 to 399 of the *Philosophical Transactions* ending with "Thus far Mr. Bayes's essay." A second observation about the manuscripts is that they are generally short, the longest on trinomial divisors being slightly more than three handwritten pages. Further, the four manuscripts relating to the derivation of $z!$ at first glance appear unrelated since they are given as self-contained results on separate sheets of paper with no notes connecting them. The manuscript on finding the infinite series expansion for $(\arcsin(z))^n$ shows Bayes leaving out significant background detail in the derivation. We also see Stanhope trying to fill in the detail for Bayes's paper on trinomial divisors. Finally, Bayes gives no motivation for considering any of the problems he worked on, nor does he provide any applications of his results. From this evidence, it is possible that Price was faced with a collection of manuscripts that contained initially seemingly unrelated results. Subsequently, Price pieced them together to form a coherent essay. For example, Section I of the paper may have been put together from a set of notes on the elements of probability that Bayes had kept. Section II of the paper related to the model table, which gives the prior distribution, would have been a separate manuscript or constructed from a set of separate manuscripts. This conjecture, however, conflicts directly with Price's own statements in the letter to Canton introducing the essay and in the layout that ends with "Thus far Mr. Bayes's essay."

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